

Conic D-branes

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The shape of D-branes is of fundamental interest in string theory. We find that, generically, D-branes in trivial spacetime can form a conic shape under external uniform forces. Surprisingly, the apex angle is found to be unique, once the spatial dimensions of the cone are given. In particular, it is universal irrespective of the external forces. The quantized angle is reminiscent of the Taylor cones of hydrodynamic electrospray. We provide explicit D-brane solutions as well as the mechanism of a force balance on the cone, for D-branes in Ramond–Ramond and Neveu–Schwarz–Neveu–Schwarz flux backgrounds. Critical embedding of probe D-branes in anti-de Sitter/conformal field theory with electric and magnetic fields is in the same category, for which we give an analytic proof of a power-law spectrum of “turbulent meson condensation”.
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Subject Index B20, B21, B22, B23

1. Introduction

Spiky branes are of fundamental interest in string theory and M-theory, since the spike singularity is an obstacle in the issue of membrane quantization [1]. For membranes, deformations of their surfaces exist, allowing thin spikes to occur without increasing the volume; the membranes then suffer from infinite degeneracies resulting in a continuous spectrum. To resolve this fundamental issue, it is necessary to discover how spiky/conic branes can be stabilized classically.

Based on this motivation, in this paper we find new conic D-brane configurations in the background flux. They are solutions of classical D-brane effective actions, which are Dirac–Born–Infeld (DBI) actions. Surprisingly, the apex angle of the cone is found to be universal and depends only on the dimensions of the cone worldvolume.

Note that the D-brane cones that we have found do not use nontrivial topologies of background spacetime in which the D-branes are embedded. D-branes wrapping conifolds or orbifolds have been studied in various contexts in string theory, but ours are not of that kind. In the target space, the tips of our D-brane cones are not located at special points (such as the tips of conifolds or black hole horizons). Our examples include a previously known probe D-brane configuration in the anti-de Sitter/conformal field theory (AdS/CFT) correspondence where a critical value of an electric field on the brane forces it to have a conic shape, which is called critical embedding [2,3].¹

¹ This critical embedding with the electric field in AdS/CFT is different from the critical embedding at thermal phase transition [4,5] where the apex of the probe D-brane touches the event horizon of a background black hole spacetime. As we have emphasized, the conic D-branes of interest to us are formed without the help of nontrivial background spacetime.

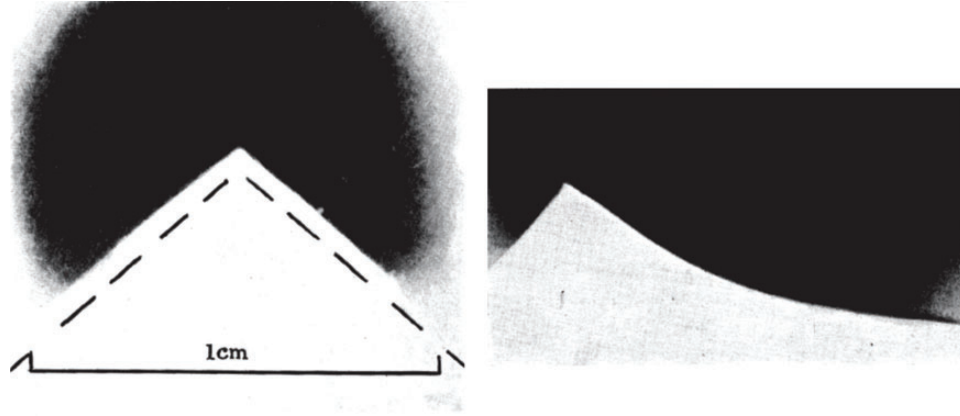


Fig. 1. Photos of Taylor cones formed at an oil/water interface in experiments [6]. Left: Pointed summit formed under an electric field in the vertical direction. Right: the whole view around the cone.

We are motivated by a hydrodynamic phenomenon called Taylor cones [6], which are widely used in electrospray in material/industrial sciences (see Fig. 1). The Taylor cones are generically formed for charged surfaces of liquids under some background electric field. The mechanism is quite simple: the induced charges on the surface of the liquid repel each other and cancel the surface tension, and the instability grows to form a cone whose apex has a vanishing tension due to the cancellation. Interestingly, Taylor cones are popular for their universal cone angle $\theta_{\text{cone}} = 49.29^\circ$, which can be easily derived from the tension balance on the surface and with the Maxwell equations. The simple reasoning of the Taylor cones leads us to the new examples of D-brane cones presented in this paper.

We have two examples of conic D-branes, which mimic the Taylor cones. The first one is a D2–D0 bound state under a constant flux of a Ramond–Ramond (RR) 1-form field. The D0-brane charges on the D2-brane are pulled by the flux, as in the Taylor cone. We solve the DBI equation of motion and find a conic solution. The second example is a Dp -brane under the constant flux of a Neveu–Schwarz–Neveu–Schwarz (NSNS) 2-form field. Induced fundamental string charges on the Dp -brane are pulled by the flux, and the Dp -brane forms a conic shape.

We can explicitly show the force balance on the conic D-branes for the two examples as well as the previously identified critical embedding of the probe D-brane with electric and magnetic fields in AdS/CFT. The force balance is quite simple: the external force has two components, one for the direction parallel to the generating line of the cone and the other for the perpendicular direction. The parallel force is canceled by the tension of the D-brane, while the perpendicular force is canceled by the surface tension of the round shape of the surface, defined by an extrinsic curvature.

Together with the example of the critical embedding of the probe D-brane in the electric field, we find that all examples share the same property: universal cone angle. It is given by

$$\theta_{\text{cone}} = \arctan \sqrt{2(d_{\text{cone}} - 1)}, \quad (1.1)$$

where d_{cone} are the spatial dimensions of the cone (including the direction of the generating line of the cone), i.e., the cone is locally $\mathbf{R}^+ \times \mathbf{S}^{d_{\text{cone}}-1}$. The “quantized” universal angle of D-brane cones is reminiscent of the Taylor cone angle.

The conic shape of the probe D-brane in the AdS/CFT example is closely related to a phase transition. The critical embedding showing the cone appears at the phase boundary of the meson melting

transition [2,3] caused by the electric field in supersymmetric quantum chromodynamics (QCD). The D-brane configuration is decomposed into radial modes that correspond to meson expectation values, and was found to exhibit a power-law spectrum [7–9]. Time-dependent simulations such as dynamically applied electric fields were performed in Refs. [7,8,10–12] and a similar power-law spectrum also appears there [7,8,12], where a singularity formation resembles the famous Bizon–Rostworowski conjecture about AdS turbulent instability [13]. All the above turbulent behavior with a power-law spectrum was confirmed numerically in both the static and time-dependent cases, but in this paper we provide an analytical proof of such power-law behavior of the static “turbulent meson condensation”, based on the conical shape of the probe D-brane.

The organization of this paper is as follows. First, in Sect. 2, we provide a generic analysis of a membrane cone under an external force, for a given stress–energy tensor on the membrane. Then, in Sect. 3, we provide three examples of conic D-brane solutions: a D2–D0 bound state in RR flux, a Dp-brane in NSNS flux, and a probe D7-brane in AdS/CFT. The first and second examples are new. Then we check the force balance for all the cases, and find that the apex angle is universally given by (1.1). In Sect. 4, we give an analytic proof of the power law for static meson turbulence on the D7-brane in AdS/CFT. We present our conclusion and discussions in the final section.

2. Membrane cone under a uniform external force

In this section, we summarize how a membrane can form a cone as a result of a force balance on the cone surface. The external force has a component perpendicular to the cone surface, which is canceled by the stress of the membrane with the extrinsic curvature as for the round shape. First, we provide an intuitive picture of the force balance, and then we provide a covariant formulation. Both lead to a specific formula of a universal angle at the apex of the cone.

2.1. Force balance in Newtonian mechanics

Consider a membrane in flat space with 3 spatial dimensions. Let us assume that the system, including the external force, is axially symmetric around the $x^3 (= y)$ axis. Then the membrane configuration is given by

$$(x^1, x^2, y) = (x^1, x^2, \phi(\rho)) \quad (2.1)$$

where $\rho \equiv \sqrt{(x^1)^2 + (x^2)^2}$. Forming a cone means the constraint

$$\frac{d\phi(\rho)}{d\rho} = \cot \theta_{\text{cone}} \quad (2.2)$$

where θ_{cone} is a constant, which is the half-cone angle (see Fig. 2). Along the radial direction of the cone, we can define a radial coordinate

$$r = \frac{\rho}{\sin \theta_{\text{cone}}}. \quad (2.3)$$

Let us consider a force balance condition. The first force balance condition is along the cone surface. The external force F_r along the cone radial direction needs to be balanced by the surface

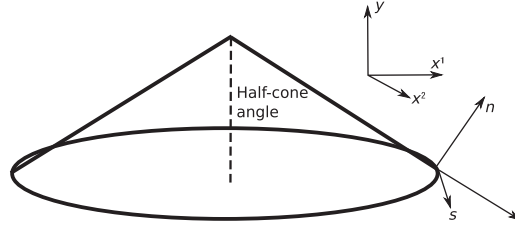


Fig. 2. The cone and our coordinate system. The cone axis is taken along y , and the half-cone angle θ_{cone} is shown. On the surface, we have two unit tangent vectors \vec{r} and \vec{s} , which are along the radial and angular directions, respectively, and a normal vector \vec{n} .

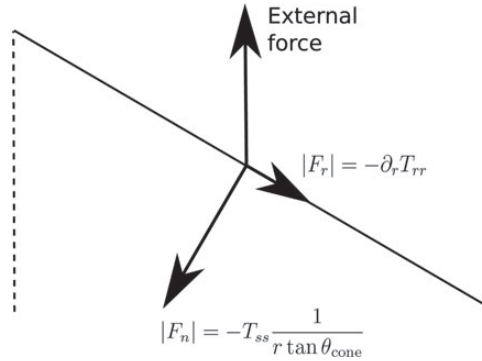


Fig. 3. The force balance at the surface.

tension as

$$|F_r| = -\partial_r T_{rr} \quad (2.4)$$

where T_{rr} is the proper rr component of the stress tensor on the conic membrane. This force, similar to a hydrodynamic equilibrium, is given by the gradient of the stress tensor.

The second force balance condition is along the direction perpendicular to the cone surface. There is a surface stress tension caused by the curvature of the membrane surface. The cone curvature is nontrivial along the circular direction of the cone. Calling the proper component of the stress tensor in the circular direction (which we call s) on a point of the cone surface T_{ss} , the Young–Laplace equation tells us that the force oriented into the cone (perpendicular to the cone surface) is balanced with the normal component of the external force as

$$|F_n| = -T_{ss} \frac{1}{r \tan \theta_{\text{cone}}}. \quad (2.5)$$

In this force balance condition, the last factor is due to the curvature. In summary, we have two force balance conditions, (2.4) and (2.5); see Fig. 3.

It is important to note that the external force can be eliminated in the force balance conditions (2.4) and (2.5), if the combined vector $\vec{F}_r + \vec{F}_n$ is oriented along the y axis. This is a natural assumption, since the cone isometry prefers the symmetry of the external force configuration. For a given cone angle θ_{cone} , the total external force directed along the y axis means

$$|F_n| = |F_r| \tan \theta_{\text{cone}}. \quad (2.6)$$

Substituting (2.4) and (2.5) into this relation, we can eliminate the strength of the external force from the force balance conditions (2.4) and (2.5) as

$$T_{ss} = (\tan \theta_{\text{cone}})^2 r \frac{d}{dr} T_{rr}. \quad (2.7)$$

Whenever this condition is satisfied for a diagonal stress tensor that depends only on r , the force balance for the uniform force directed along the axis of the cone is ensured.

Now, let us assume that the membrane has only isotropic stress as

$$T_{ss} = T_{rr}. \quad (2.8)$$

The stress tensor can be approximated as a power function near the apex of the cone $r \sim 0$; thus, we simply assume

$$T_{ss} = T_{rr} \simeq A r^\alpha \quad (r \sim 0) \quad (2.9)$$

where A and α are constant parameters. Then the relation of the stress tensor components (2.7) can be easily solved as

$$\theta_{\text{cone}} = \arctan \sqrt{\frac{1}{\alpha}}. \quad (2.10)$$

Therefore, the membrane dynamics completely determines the cone angle θ_{cone} , and, furthermore, the cone angle depends only on how the stress tensor behaves near the apex (2.9).

Generically, the energy–momentum tensor on the membrane depends on the properties of the material, such as what kind of equation of state the membrane has, how the membrane behaves under the external force, and so on. For example, the charge distribution on the membrane is essential for the energy–momentum tensor, while the distribution normally depends on the binding energy and the repulsive forces between the charges.

Let us summarize what we have described here. We have assumed the following three things: (1) The system is axially symmetric in 3 spatial dimensions, (2) the force is along the axis of symmetry, and (3) the stress on the brane is isotropic. Then it follows that the cone angle θ_{cone} is given universally as (2.10). It depends only on the radial power α of the stress (2.9). In particular, when the cone is formed, the stress at the cone apex vanishes. In other words, when the cone apex stress does not vanish, no cone is formed.

2.2. Covariant treatment

In this subsection, we shall provide a covariant formulation for the dynamics of membranes in general curved spacetimes to see the force balance for various conic membranes. Let $\{x^\mu\}$ and $\{y^a\}$ be coordinates on the bulk spacetime and the worldvolume of a membrane, respectively. When the embedding of the membrane in the bulk spacetime is determined by $x^\mu = X^\mu(y^a)$, the induced metric on the membrane is given by

$$h_{ab} = g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu, \quad (2.11)$$

where $g_{\mu\nu}$ denotes the bulk spacetime metric. For later convenience, we define a projection tensor mapping from vectors in the bulk to vectors on the membrane as $h_a{}^\mu \equiv \partial_a X^\mu$.

It is known that the extrinsic and intrinsic dynamics of the membrane are governed by the equations simplified geometrically as

$$T^{ab} K^\mu_{ab} = -\mathcal{F}_n^\mu, \quad (2.12)$$

$$D_a T^{ab} = \mathcal{F}_t^b, \quad (2.13)$$

where T^{ab} is the stress–energy tensor of the membrane and D_a denotes the covariant derivative with respect to the induced metric h_{ab} . K^μ_{ab} is the extrinsic curvature, which is given by²

$$\begin{aligned} -K^\mu_{ab} &\equiv (g^\mu_\lambda - h^\mu_\lambda) h_a^\nu \nabla_\nu h_b^\lambda \\ &= D_a D_b X^\mu + \Gamma^\mu_{\alpha\beta} D_a X^\alpha D_b X^\beta, \end{aligned} \quad (2.14)$$

where $\Gamma^\mu_{\alpha\beta}$ is the Christoffel symbol for the bulk metric. \mathcal{F}_n^μ and \mathcal{F}_t^a are normal and tangential components of external forces acting on some charges with the membrane, which are defined by $\mathcal{F}^\mu = \mathcal{F}_n^\mu + \mathcal{F}_t^a h_a^\mu$ and $g_{\mu\nu} h_a^\mu \mathcal{F}_n^\nu = 0$. We note that Eq. (2.12) corresponds to Eq. (2.5) in the previous subsection and Eq. (2.13) corresponds to Eq. (2.4), which yields fundamental equations of hydrodynamics or elastic dynamics.

Now, we suppose that an “axisymmetric” bulk spacetime is given by

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= \mathbf{A}_{ij}(\rho, \zeta) dy^i dy^j + \mathbf{B}(\rho, \zeta) (d\rho^2 + d\zeta^2) + \mathbf{C}(\rho, \zeta) d\Omega_{d-1}^2 \\ &\quad + \mathbf{D}_{kl}(\rho, \zeta) dw^k dw^l, \end{aligned} \quad (2.15)$$

where ρ and ζ can be interpreted as radial and horizontal coordinates in the usual cylindrical coordinate system. Note that we assume

$$\partial \mathbf{A}_{ij} / \partial \rho|_{\rho=0} = 0, \quad (2\sqrt{\mathbf{B}\mathbf{C}})^{-1} \partial \mathbf{C} / \partial \rho|_{\rho=0} = 1, \quad \mathbf{C}|_{\rho=0} = 0 \quad (2.16)$$

for the bulk spacetime to be regular at the axis $\rho = 0$. When a membrane is axisymmetrically embedded by functions $\zeta = \phi(\rho)$ and $w^k = \text{const.}$, the induced metric on the membrane becomes

$$\begin{aligned} h_{ab} dy^a dy^b &= \mathbf{A}_{ij}(\rho, \phi(\rho)) dy^i dy^j + [1 + \phi'(\rho)^2] \mathbf{B}(\rho, \phi(\rho)) d\rho^2 \\ &\quad + \mathbf{C}(\rho, \phi(\rho)) d\Omega_{d-1}^2. \end{aligned} \quad (2.17)$$

By introducing $\sin \theta(\rho) \equiv 1/\sqrt{1 + \phi'^2}$ and $\cos \theta(\rho) \equiv \phi'/\sqrt{1 + \phi'^2}$, the unit normal 1-form and vector along the nontrivial normal direction for the membrane are

$$n_\mu dx^\mu = \sqrt{\mathbf{B}} (\cos \theta d\rho - \sin \theta d\zeta), \quad n^\mu \partial_\mu = \frac{1}{\sqrt{\mathbf{B}}} (\cos \theta \partial_\rho - \sin \theta \partial_\zeta). \quad (2.18)$$

On the other hand, the unit tangent 1-form and vector along the radial direction (namely, the generatrix of the cone) are

$$r_\mu dx^\mu = \sqrt{\mathbf{B}} (\sin \theta d\rho + \cos \theta d\zeta), \quad r^\mu \partial_\mu = \frac{1}{\sqrt{\mathbf{B}}} (\sin \theta \partial_\rho + \cos \theta \partial_\zeta). \quad (2.19)$$

Note that on the membrane these 1-form and vector can be written as $r_a dy^a = \sqrt{\mathbf{B}}/\sin \theta d\rho$ and $r^a \partial_a = \sin \theta / \sqrt{\mathbf{B}} \partial_\rho$.

We assume that the stress–energy tensor of the membrane has the following form:

$$T_{ab} = \tau_{ab} - \sigma (r_a r_b + s_{ab}), \quad (2.20)$$

where s_{ab} is the spherical part of the induced metric, i.e., $s_{ab} dy^a dy^b = \mathbf{C} d\Omega_{d-1}^2$. Since τ_{ab} are com-

² Since $\partial_a = h_a^\mu \partial_\mu$ are a coordinate basis on the submanifold representing the membrane, ∂_a and ∂_b commute. This means that $[\partial_a, \partial_b]^\mu = h_a^\nu \nabla_\nu h_b^\mu - h_b^\nu \nabla_\nu h_a^\mu = 0$ and K^μ_{ab} is symmetric in a and b .

ponents other than those on the cone, $\tau_{ab}r^b = 0$ and $\tau_{abs}{}^{bc} = 0$ are satisfied. If $\sigma > 0$, it means that the membrane has isotropic tension on the cone.³

For the normal direction along n^μ , from Eq. (2.12) we have

$$\begin{aligned}\mathcal{F}^\mu n_\mu &= -T^{ab} K_{ab} \\ &= -\tau^{ab} K_{ab} + \sigma K_{ab} r^a r^b + \sigma s^{ab} K_{ab},\end{aligned}\quad (2.21)$$

where $n_\mu K^\mu{}_{ab} = K_{ab} \equiv h_a{}^\mu h_b{}^\nu \nabla_\mu n_\nu$. For the tangential direction along r^μ , from Eq. (2.13) we have

$$\begin{aligned}\mathcal{F}^\mu r_\mu &= r_b D_a T^{ab} \\ &= -D_a (\sigma r^a) + \sigma s^{ab} D_a r_b - \tau^{ab} D_a r_b.\end{aligned}\quad (2.22)$$

If the external force is along the axis of the cone, namely, only \mathcal{F}^ζ is a non-zero component,⁴ then we obtain $\mathcal{F}^\mu (r_\mu \sin \theta + n_\mu \cos \theta) = 0$. Combining Eqs. (2.21) and (2.22) yields

$$-\sin \theta D_a (\sigma r^a) + \sigma s^{ab} \frac{1}{2\sqrt{B}} \partial_\rho s_{ab} - \tau^{ab} \frac{1}{2\sqrt{B}} \partial_\rho A_{ab} + \cos \theta \sigma r^a r^b K_{ab} = 0, \quad (2.23)$$

where we have used

$$\frac{1}{\sqrt{B}} \frac{\partial}{\partial \rho} = \sin \theta r^\mu \partial_\mu + \cos \theta n^\mu \partial_\mu. \quad (2.24)$$

By using the metric functions explicitly, this can be written as

$$\frac{\sin \theta}{\sqrt{-AB}} \frac{d}{d\rho} \left(\sqrt{-A} \sigma \sin \theta \right) - \sigma \cos \theta n^\mu \partial_\mu \log \left(\sqrt{B} C^{(d-1)/2} \right) + \frac{1}{2\sqrt{B}} \tau^{ij} \partial_\rho A_{ij} = 0, \quad (2.25)$$

where $A \equiv \det A_{ij}$.

Since we have assumed that the spacetime is regular at $\rho = 0$ by imposing the regularity condition (2.16), the metric function can be expanded around $\rho = 0$ as

$$A_{ij} = A_{0ij}(\zeta) + \mathcal{O}(\rho^2), \quad B = B_0(\zeta) + \mathcal{O}(\rho^2), \quad C = B_0(\zeta) \rho^2 + \mathcal{O}(\rho^4), \quad (2.26)$$

where a regular polar coordinate needs $B_0 \neq 0$. In addition, we assume that $\det A_{0ij} \neq 0$ on the membrane, which means that the membrane does not touch Killing horizons in the bulk.⁵ If we assume in (2.20) that the tension σ plays a dominant role, $\sigma \gg |\tau| \rho^2$ for $\rho \sim 0$, we have the following condition:

$$(\sin \theta_{\text{cone}})^2 \frac{d\sigma}{d\rho} \simeq (d-1) (\cos \theta_{\text{cone}})^2 \frac{\sigma}{\rho}, \quad (2.27)$$

where $\theta_{\text{cone}} \equiv \theta(\rho = 0)$. Now, supposing that the tension behaves as $\sigma \sim \rho^\alpha$ ($\alpha > 0$) near the apex of cone $\rho \sim 0$, then the angle of the cone becomes

$$\theta_{\text{cone}} = \arctan \sqrt{\frac{d-1}{\alpha}}. \quad (2.28)$$

This is our formula for the cone angle, simply written by only the cone dimension and the scaling of the tension.

³ Note that for Nambu–Goto branes the energy density is equal to the tension, but in general they are different. In Appendix A, we study the stress–energy tensor for a system described by Dirac–Born–Infeld (DBI) action (see (A7)).

⁴ Here, we consider that both the external force and the normal vector are in the direction of the same side for the membrane, namely, $\mathcal{F}^\mu n_\mu > 0$.

⁵ On the other hand, if the membrane touches bulk black holes, such as the critical embedding at thermal phase transition, $\sqrt{-A} \sim 0$ in Eq. (2.25) plays a significant role. Physically, this means that the infinite gravitational blueshift near the horizon becomes more significant than the matter distribution.

Obviously, the cone angle formula (2.28) generalizes the previous formula (2.10) in the following respects: (1) it works in a general geometry, (2) it uses a generic energy–momentum tensor on the brane, and (3) the brane can have arbitrary worldvolume dimensions. In the next section, we study examples of D-branes in string theory, and will find that $\alpha = 1/2$ for every example we consider.

3. Conic D-branes and universal cone angle

The phenomenon of the Taylor cones suggests that D-branes in superstring theory can develop a conic shape under some background field. Furthermore, it suggests the existence of a universal cone angle, as the Taylor cones are formed by quite a simple mechanism that can be generalized to higher dimensions.

In this section, we provide three new examples of conic D-branes (Sects. 3.1, 3.2, 3.3), and provide a universal formula for the cone angle that just depends on the dimensions of the cone worldvolume (Sect. 3.4).

The Taylor cone is formed simply because ion charges on the liquid surface are pulled by a background electric field and cancel the liquid surface tension. So it is natural to expect higher-dimensional analogues in string theory. The three examples of the background field that we present here are: The case of a Ramond–Ramond (RR) flux, the case of a Neveu–Schwarz–Neveu–Schwarz (NSNS) flux, and the case of AdS/CFT with electric and magnetic fields.

3.1. D-brane cone in RR flux

3.1.1. Conic solution

The first example is a D-brane in a constant RR flux background in flat spacetime. The D-brane is an analogue of the membrane-like surface of the Taylor cones. To put “ions” on the membrane, we consider lower-dimensional D-branes on the D-brane. To have a bound state, it is suitable to choose $D(p-2)$ -branes bound on a single Dp -brane. Then we turn on a background RR flux, which pulls the bound $D(p-2)$ -branes. The RR gauge field should be a $(p-1)$ -form field for which the $D(p-2)$ -branes are charged. Below, we shall show that the conic shape of the Dp -brane can be obtained as a classical solution of the Dp -brane worldvolume effective action.

We can choose $p = 2$ without losing generality, as all other choices of p are obtained by T-dualities. The bound D0-branes are described by a field strength of the D-brane gauge field. The D2-brane effective action is a Dirac–Born–Infeld (DBI) action plus a coupling to the RR field:

$$S = -\mathcal{T}_2 \int dx^0 dx^1 dx^2 \sqrt{-\det(\eta_{ab} + 2\pi\alpha' F_{ab} + \partial_a \phi \partial_b \phi)} - \frac{\mathcal{T}_2}{2} \int dx^0 dx^1 dx^2 2\pi\alpha' F_{ab} C_c \epsilon^{abc}. \quad (3.1)$$

Here, $a, b = 0, 1, 2$ are the worldvolume directions of the D2-brane, and F_{ab} is the gauge field strength on the D2-brane. C_c is the RR 1-form in the bulk space. ϕ is a scalar field on the D-brane that measures the displacement of the position of the D2-brane in a direction transverse to the D2-brane worldvolume. We choose one direction ϕ among 7 transverse directions for simplicity.

We turn on the temporal component of the RR background

$$C_0 = C_0[\phi], \quad (3.2)$$

where $C_0[\phi]$ is an arbitrary functional of $\phi(x)$. Having this is indeed equivalent to having a nontrivial background RR field strength $H_{\phi 0}^{\text{RR}} \equiv \partial_\phi C_0[\phi]$.

We are interested in a static D2-brane configuration without any electric field on it, so the action can be simplified⁶ with $F_{01} = F_{02} = \partial_0\phi = 0$. We obtain the action as

$$S = \text{const.} \int dx^0 dx^1 dx^2 \left[-\sqrt{1 + (2\pi\alpha' F_{12})^2 + (\partial_1\phi)^2 + (\partial_2\phi)^2} + 2\pi\alpha' C_0[\phi] F_{12} \right]. \quad (3.3)$$

Here the overall normalization of the RR field C_0 was chosen to simplify the Lagrangian.⁷

The equations of motion are

$$\partial_i \left[\frac{\partial_i \phi}{\sqrt{1 + (2\pi\alpha' F_{12})^2 + (\partial_1\phi)^2 + (\partial_2\phi)^2}} \right] + 2\pi\alpha' \frac{dC_0[\phi]}{d\phi} F_{12} = 0, \quad (3.4)$$

$$\partial_i \left[\frac{2\pi\alpha' F_{12}}{\sqrt{1 + (2\pi\alpha' F_{12})^2 + (\partial_1\phi)^2 + (\partial_2\phi)^2}} \right] - \partial_i C_0[\phi] = 0, \quad (3.5)$$

where $i = 1, 2$. The second equation can be integrated to give

$$2\pi\alpha' F_{12} = C_0[\phi] \sqrt{\frac{1 + (\partial_i\phi)^2}{1 - (C_0[\phi])^2}}. \quad (3.6)$$

Here we absorbed the integration constant to a redefinition of $C_0[\phi]$ by a constant shift ($C_0[\phi] \rightarrow C_0[\phi] + \text{const.}$) without losing the generality, and chose a sign for our later convenience. It is amusing that (3.6) is a first-order differential equation, so we may call (3.6) a “BPS” (Bogomol’nyi–Prasad–Sommerfield) equation.

Substituting (3.6) into (3.4), we obtain a differential equation for ϕ :

$$\partial_i \left[\frac{\partial_i \phi}{\sqrt{1 + (\partial_j\phi)^2}} \sqrt{1 - (C_0[\phi])^2} \right] + \frac{dC_0[\phi]}{d\phi} C_0[\phi] \sqrt{\frac{1 + (\partial_j\phi)^2}{1 - (C_0[\phi])^2}} = 0. \quad (3.7)$$

The equation is singular when $C_0[\phi] = \pm 1$. In fact, we will see that this point in the bulk provides the tip of the cone solution. It is instructive to note that Eq. (3.7) can be derived from an “effective” action:

$$S_{\text{cone}} = \int dx^1 dx^2 \sqrt{(1 + (\partial_i\phi)^2) (1 - (C_0[\phi])^2)}. \quad (3.8)$$

We consider a typical RR background, i.e., a constant RR field strength along the direction ϕ , as

$$\partial_\phi C_0 = c \quad (3.9)$$

where $c (>0)$ is a constant parameter. Up to a gauge choice we can take

$$C_0 = c \phi. \quad (3.10)$$

This background can be thought of as a local approximation of a generic $C_0[\phi]$ background. In fact, to show the existence of conic D-branes, the local approximation is enough. For this constant RR flux

⁶ The equation of motion for A_0 (which is Gauss’s law) is satisfied by $F_{01} = F_{02} = \partial_0\phi = 0$ because the action is quadratic in these components.

⁷ $\epsilon_{012} = +1$ and $\epsilon^{012} = -1$ in our convention.

background, the singularity in Eq. (3.7) is found at $\phi = 1/c$. We assume rotational symmetry for the solutions, and expand ϕ around this singular value,

$$\phi = 1/c - f(\rho), \quad (3.11)$$

where $\rho \equiv \sqrt{(x^1)^2 + (x^2)^2}$ is the radial coordinate on the D2-brane worldvolume. Then, to the leading order in $f(\rho)$, we obtain an equation for $f(\rho)$ as

$$\frac{1}{\rho} \partial_\rho \left[\rho \frac{\partial_\rho f}{\sqrt{1 + (\partial_\rho f)^2}} \sqrt{f} \right] - \frac{1}{2\sqrt{f}} \sqrt{1 + (\partial_\rho f)^2} = 0. \quad (3.12)$$

Again, the equation can be obtained from an “effective” 1D action:

$$S_{\text{cone}} = \int d\rho \rho \sqrt{f(\rho) \left(1 + (\partial_\rho f(\rho))^2 \right)}. \quad (3.13)$$

We will find later that this effective action can be universally found in string theory.

To solve Eq. (3.12), we consider the following ansatz:

$$\phi = \frac{1}{c} - a\rho^b \quad (b > 0), \quad (3.14)$$

which goes to $\phi = 1/c$ at $\rho = 0$. Substituting this into (3.12), we can show that any nontrivial solution has a unique form:

$$a = \frac{1}{\sqrt{2}} \quad b = 1. \quad (3.15)$$

The value $b = 1$ indeed shows a cone, as the radius of the section of the D2-brane at fixed ϕ is given by a linear function of ρ . So, finally, we can show that the D2-brane configuration reaching $\phi = 1/c$ is conic:

$$\phi = \frac{1}{c} - \frac{1}{\sqrt{2}}\rho + \text{higher}. \quad (3.16)$$

We have two comments on the conic D-brane configuration. First, let us evaluate the D0-brane charge density. Substituting the near-tip solution into (3.6), we find

$$2\pi\alpha' F_{12} = 3^{1/2} 2^{-3/4} c^{-1/2} \frac{1}{\sqrt{\rho}} \quad (3.17)$$

near $\rho = 0$. This shows that the bound D0-brane charge density is proportional to $1/\sqrt{l}$ where l is the distance from the tip of the cone. On the other hand, it is known that the Taylor cone has a charge distribution $\sim 1/\sqrt{l}$ around the tip of the cone. So, our result is similar to the Taylor cone.

The second comment is about the asymptotics. Solutions of the full equation of motion (3.7) for $C_0[\phi] = c\phi$ are shown in Figs 4 and 5. They indicate that, even at large ρ , the shape follows that of a cone. However, this is not that case. As seen from Eq. (3.7), the configuration of ϕ is limited to the region

$$-\frac{1}{c} < \phi(\rho) < \frac{1}{c}. \quad (3.18)$$

Otherwise Eq. (3.6) provides an imaginary field strength F_{12} . So, the brane configuration terminates when it reaches $\phi = -1/c$. This strange behavior is due to our approximation of constant RR field strength. Generically, in string theory, the RR field varies in space, and our constant approximation is valid only locally. The asymptotics depends on the global structure of the RR background.

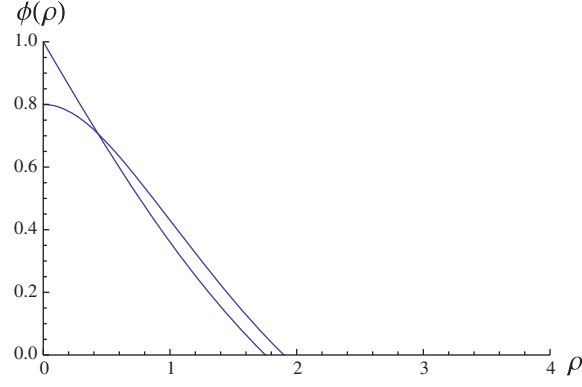


Fig. 4. A numerical solution $\phi(\rho)$ for the full equation of motion (3.7) with $C_0[\phi] = c\phi$. The straight line is for $\phi(0) = 1/c$, which is the “singular” point, and the curved line is for $\phi(0) = 4/5c$, which is below the singular point. We took $c = 1$ for this plot. We find that a cone is formed for the case $\phi(\rho = 0) = 1/c$.

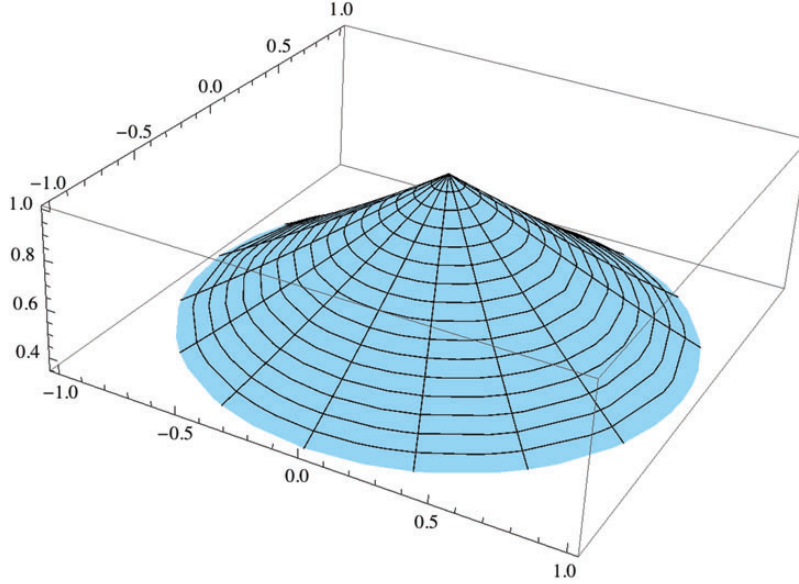


Fig. 5. A view of the conic D2-brane. The parameters are the same as those in the previous figure.

3.1.2. Force balance of the cone

Let us follow the argument in Sect. 2.2 about a covariant treatment of the force balance, for this conic D-brane in an RR flux. We can explicitly see that the force balance condition provides the conic solution (3.16).

We consider the D2-brane described by the embedding function $y = \phi(x^1, x^2)$ in the RR background $C_0 = cy$ in flat spacetime:

$$g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + (dx^1)^2 + (dx^2)^2 + d\vec{y}_{d-3}^2. \quad (3.19)$$

Assuming it to be static and axisymmetric, the induced metric on the D2-brane is

$$h_{ab} dx^a dx^b = -dt^2 + (1 + \phi'^2) d\rho^2 + \rho^2 d\theta^2, \quad (3.20)$$

where $\rho^2 = (x^1)^2 + (x^2)^2$. The unit vector and 1-form normal to the brane are

$$n^\mu \partial_\mu = \frac{1}{\sqrt{1+\phi'^2}} (\phi' \partial_\rho - \partial_y), \quad n_\mu dx^\mu = \frac{1}{\sqrt{1+\phi'^2}} (\phi' d\rho - dy). \quad (3.21)$$

The unit vector and 1-form tangent to the brane are

$$r^\mu \partial_\mu = \frac{1}{\sqrt{1+\phi'^2}} (\partial_\rho + \phi' \partial_y), \quad r_\mu dx^\mu = \frac{1}{\sqrt{1+\phi'^2}} (d\rho + \phi' dy). \quad (3.22)$$

We note that, strictly speaking, since there are many codimensions, other directions normal to the brane exist. However, we can focus on only the above normal vector n because for the other directions the brane is trivially embedded. The non-vanishing components of the extrinsic curvature $n_\mu K^\mu_{ab} = K_{ab} \equiv \frac{1}{2} \mathcal{L}_n h_{ab}$ are

$$K_{\mu\nu} r^\mu r^\nu = \left(\frac{\phi'}{\sqrt{1+\phi'^2}} \right)', \quad K^\theta_\theta = \frac{\phi'}{\sqrt{1+\phi'^2}} \frac{1}{\rho}. \quad (3.23)$$

By using

$$2\pi\alpha' F_{12} = C_0[\phi] \sqrt{\frac{1+\phi'^2}{1-(C_0[\phi])^2}} \quad (3.24)$$

from Eq. (3.6), the non-vanishing components of the stress–energy tensor are

$$T^0_0 = -\sqrt{\frac{1+\phi'^2 + (2\pi\alpha' F_{12})^2}{1+\phi'^2}} = -\frac{1}{\sqrt{1-(C_0[\phi])^2}},$$

$$T^\theta_\theta = T_{\mu\nu} r^\mu r^\nu = -\sqrt{\frac{1+\phi'^2}{1+\phi'^2 + (2\pi\alpha' F_{12})^2}} = -\sqrt{1-(C_0[\phi])^2}. \quad (3.25)$$

It turns out that the isotropic tension $\sigma = -T^\theta_\theta = -T_{\mu\nu} r^\mu r^\nu$ defined in Eq. (2.20) is now given by $\sigma = \sqrt{1-(C_0[\phi])^2}$, which will vanish when $C_0[\phi] = \pm 1$.

The equation of motion is alternatively written as

$$T_{\mu\nu} r^\mu r^\nu K_{\alpha\beta} r^\alpha r^\beta + T^\theta_\theta K^\theta_\theta = -n_\mu \mathcal{F}^\mu, \quad (3.26)$$

which is nothing but the extrinsic force balance (2.12) in Sect. 2.2. In this case, the external force is given by $\mathcal{F}^\mu = -\frac{1}{2} F_{ab} e^{abc} H^\mu_c$. If we suppose that, when the tension vanishes at $C_0[\phi]^2 = 1$, the brane becomes conical, namely, $\rho = 0$, the equation of motion reduces to

$$n_\mu \mathcal{F}^\mu \simeq -T^\theta_\theta K^\theta_\theta. \quad (3.27)$$

This can be rewritten as

$$-\frac{1}{\sqrt{1+\phi'^2}} \frac{C_0}{\sqrt{1-C_0^2}} \frac{dC_0}{d\phi} \simeq \sqrt{1-C_0^2} \frac{\phi'}{\sqrt{1+\phi'^2}} \frac{1}{\rho}. \quad (3.28)$$

Solving the force balance equation tells us that

$$\phi'|_{\rho=0} = -1/\sqrt{2}. \quad (3.29)$$

This is nothing but the conic solution (3.16). Here we have found that the force balance condition is nothing but the equations of motion of the D-brane. In other words, the solution of the equations of motion should necessarily satisfy the force balance condition.

3.2. D-brane cone in NSNS flux

3.2.1. Conic solution

The next example is a D-brane in an NSNS flux background in flat spacetime. The charged bound object this time is fundamental strings, which couple to the NSNS gauge field in the bulk. Interestingly, the NSNS flux in the background induces a fundamental string charge on the D-brane, so this time we do not need to prepare for the charged object on the D-brane in the first place, as we will see below. This situation is in contrast to the previous D2D0 case, where we needed F_{12} on the D2-brane.

Let us consider a single D p -brane in a constant background NSNS flux $H_{01\phi} = c$. (The constancy is again only for simplicity, and one can explore the full configuration once the explicit NSNS flux is given.) Here ϕ is a direction transverse to the D-brane, and x^1 is along the D-brane worldvolume. We shall show the existence of the cone configuration of the D-brane.

We choose a natural gauge for the NSNS flux,

$$B_{01} = c\phi, \quad (3.30)$$

and assume that the D-brane shape depends only on the D-brane worldvolume scalar field

$$\phi = \phi(\rho) \quad (3.31)$$

where ρ is the radial coordinate on the D-brane worldvolume except for the x^1 direction, $\rho \equiv \sqrt{(x^2)^2 + \cdots + (x^p)^2}$. The D-brane effective action is given by

$$\begin{aligned} S &= -\mathcal{T}_p \int dx^{p+1} x \sqrt{-\det(\eta + B + \partial\phi \partial\phi)} \\ &= -\mathcal{T}_p \int dx^0 dx^1 d\rho V_{p-2} \rho^{p-2} \sqrt{(1 + (d\phi/d\rho)^2) (1 - B_{01}^2)} \end{aligned} \quad (3.32)$$

where V_{p-2} is the volume of the unit $(p-2)$ -dimensional sphere. Now, we substitute (3.30) and notice that the Lagrangian density vanishes at the point $\phi = 1/c$ in the bulk spacetime. For the region $\phi > 1/c$ in the bulk spacetime, the D-brane action becomes imaginary.

We shall show that, for the D-brane to reach this “singular surface” in the bulk, the D-brane shape becomes conical. Let us expand the D-brane scalar field around this singular surface as

$$\phi(\rho) = \frac{1}{c} - f(\rho), \quad (3.33)$$

where $f(\rho = 0) = 0$. We are interested in the region close to the singular surface, so we assume that $f(\rho)$ is small, and substitute it into the action (3.32). Then, at the leading order in $f(\rho)$, we obtain

$$S = -\mathcal{T}_p \int dx^0 dx^1 d\rho V_{p-2} \rho^{p-2} \sqrt{2f(1 + (f')^2)}. \quad (3.34)$$

The equation of motion for the scalar field is

$$\frac{d}{d\rho} \left(\rho^{p-2} \frac{f'}{\sqrt{1 + (f')^2}} \sqrt{f} \right) - \rho^{p-2} \frac{1}{2\sqrt{f}} \sqrt{1 + (f')^2} = 0. \quad (3.35)$$

Near the origin $\rho = 0$, the function $f(\rho)$ can be approximated by

$$f(\rho) = a\rho^b, \quad a > 0, \quad b > 0. \quad (3.36)$$

The equation of motion (3.35) can be easily shown to have a nontrivial solution only for $b = 1$, and we find

$$a = \frac{1}{\sqrt{2p-4}}, \quad b = 1. \quad (3.37)$$

This unique solution corresponds to the D-brane configuration near $\rho = 0$,

$$\phi(\rho) = \frac{1}{c} - \frac{1}{\sqrt{2p-4}}\rho + \text{higher in } \rho. \quad (3.38)$$

We find that the D-brane forms a cone.

3.2.2. Force balance of the cone

Again, we shall investigate the force balance condition for this D-brane cone in the NSNS flux, and will see that the condition is met with the equations of motion.

We consider $(d+1)$ -dimensional flat spacetime,

$$\begin{aligned} g_{\mu\nu} dx^\mu dx^\nu &= -dt^2 + d\vec{x}_p^2 + d\vec{y}_{d-p}^2 \\ &= -dt^2 + dx_1^2 + d\rho^2 + \rho^2 d\Omega_{p-2}^2 + d\vec{y}_{d-p}^2, \end{aligned} \quad (3.39)$$

where $\rho^2 = x_2^2 + \cdots + x_p^2$. Assuming that the NSNS field is

$$B_{\mu\nu} dx^\mu \wedge dx^\nu = 2cy dt \wedge dx_1, \quad (3.40)$$

we have the field strength $H_{\mu\alpha\beta} = \partial_\mu B_{\alpha\beta} + \partial_\alpha B_{\beta\mu} + \partial_\beta B_{\mu\alpha}$ as

$$H_{01y} = c. \quad (3.41)$$

When the embedding function for the brane is given by $y = \phi(\rho)$, the induced metric is

$$h_{ab} dx^a dx^b = -dt^2 + dx_1^2 + (1 + \phi'^2) d\rho^2 + \rho^2 d\Omega_{p-2}^2, \quad (3.42)$$

and the effective metric $\gamma_{ab} \equiv h_{ab} + B_{ac} B_{bd} h^{cd}$ is

$$\gamma_{ab} dx^a dx^b = -\left(1 - c^2 \phi^2\right) dt^2 + \left(1 - c^2 \phi^2\right) dx_1^2 + \left(1 + \phi'^2\right) d\rho^2 + \rho^2 d\Omega_{p-2}^2. \quad (3.43)$$

The unit vector and 1-form normal to the brane are

$$n^\mu \partial_\mu = \frac{1}{\sqrt{1 + \phi'^2}} (\phi' \partial_\rho - \partial_y), \quad n_\mu dx^\mu = \frac{1}{\sqrt{1 + \phi'^2}} (\phi' d\rho - dy). \quad (3.44)$$

Also, those tangent to the brane are

$$r^\mu \partial_\mu = \frac{1}{\sqrt{1 + \phi'^2}} (\partial_\rho + \phi' \partial_y), \quad r_\mu dx^\mu = \frac{1}{\sqrt{1 + \phi'^2}} (d\rho + \phi' dy). \quad (3.45)$$

The non-zero components of the extrinsic curvature are

$$K_{\mu\nu} r^\mu r^\nu = \left(\frac{\phi'}{\sqrt{1 + \phi'^2}} \right)', \quad K^m_n = \frac{\phi'}{\rho \sqrt{1 + \phi'^2}} \delta^m_n, \quad (3.46)$$

where m, n are running on the $(p-2)$ sphere. The stress-energy tensor of the brane is

$$T^0_0 = T^1_1 = -\frac{1}{\sqrt{1 - c^2 \phi^2}}, \quad T_{\mu\nu} r^\mu r^\nu = -\sqrt{1 - c^2 \phi^2}, \quad T^m_n = -\sqrt{1 - c^2 \phi^2} \delta^m_n. \quad (3.47)$$

It turns out that the tension on the cone, which is given by $\sigma \equiv \sqrt{1 - c^2 \phi^2}$, becomes isotropic.

Let us write the force balance explicitly. In this case there is an external force because of the NSNS field. The external force \mathcal{F}^μ is given by

$$\mathcal{F}^\mu = \frac{1}{2} H^\mu{}_{ab} J^{ab}, \quad J^{ab} \equiv -\sigma \gamma^{ac} h^{bd} B_{cd}, \quad (3.48)$$

where J^{ab} denotes the current coupled with the NSNS field strength. The force balance along the normal direction (2.12) is written as

$$T^{\mu\nu} r_\mu r_\nu K_{\alpha\beta} r^\alpha r^\beta + T^m{}_n K^n{}_m - \frac{1}{2} \sigma n^\mu H_{\mu\alpha\beta} \partial_a X^\alpha \partial_b X^\beta \gamma^{ac} h^{bd} B_{cd} = 0. \quad (3.49)$$

This can be rewritten as

$$-\sqrt{1-c^2\phi^2} \left(\frac{\phi'}{\sqrt{1+\phi'^2}} \right)' - (p-2) \sqrt{1-c^2\phi^2} \frac{\phi'}{\rho\sqrt{1+\phi'^2}} - \frac{c^2\phi}{\sqrt{1+\phi'^2}} \frac{1}{\sqrt{1-c^2\phi^2}} = 0. \quad (3.50)$$

It turns out that this equation is nothing but the equation of motion of the D-brane.

Assuming that near $\rho \sim 0$ the brane becomes conical as $\phi(\rho) - c^{-1} \sim \rho$ and the tension σ vanishes, the equation of motion can be reduced to

$$(p-2) \sqrt{1-c^2\phi^2} \frac{\phi'}{\rho\sqrt{1+\phi'^2}} \simeq -\frac{c^2\phi}{\sqrt{1+\phi'^2}} \frac{1}{\sqrt{1-c^2\phi^2}}. \quad (3.51)$$

As a result, we have

$$2(p-2)\phi'^2|_{\rho=0} = 1, \quad (3.52)$$

which is the solution we found before, (3.38).

3.3. Probe brane cone in AdS/CFT

3.3.1. Conic solution

We shall see that the universality of the cone angle θ_{cone} is quite broadly found, and here we demonstrate a calculation in a popular setup in AdS/CFT. It is known [4,5] that there exists a “critical embedding” at which a flavor D-brane in the AdS/CFT correspondence exhibits a conical shape, which serves as a phase boundary. In the following, we will find that the cone angle takes the universal form (3.76) for a conic D7-brane with electromagnetic field in the background of AdS₅–Schwarzschild \times S⁵ geometry.

The background metric with a generic temperature T is given by

$$ds^2 = \frac{1}{R^2} \frac{w^4 + r_H^4/4}{w^2} \left[-h(w) dt^2 + dx_i^2 \right] + \frac{R^2}{w^2} \left(dw^2 + w^2 d\Omega_5^2 \right), \quad (3.53)$$

where $i = 1, 2, 3$, and we define

$$h(w) \equiv \left(\frac{w^4 - r_H^4/4}{w^4 + r_H^4/4} \right)^2. \quad (3.54)$$

The temperature is related to the usual horizon radius r_H as

$$r_H = \pi T R^2 = \pi T \sqrt{2\lambda\alpha'}. \quad (3.55)$$

At $T = 0$, the geometry reduces to that of AdS₅ \times S⁵.

We consider a D7-brane probe action in this geometry, with a generic constant electromagnetic field strength on the brane. The action in the geometry is

$$S = -\mathcal{T}_p \int d^{p+1}x \sqrt{-\det(g_{ab} + 2\pi\alpha' F_{ab} + \partial_a\phi^k \partial_b\phi^l g_{kl})} \quad (3.56)$$

where $a, b = 0, 1, \dots, 7$ are for the D7-brane worldvolume coordinates, and $k, l = 8, 9$ are transverse coordinates. We define the transverse coordinates as

$$w^2 = \rho^2 + L(\rho)^2 \quad (3.57)$$

where $\rho^2 = (x^4)^2 + \dots + (x^7)^2$. The function $L(\rho)$ describes the shape of the D7-brane in the geometry, and we assume a spherical symmetry with respect to the radial coordinate ρ on the D7-brane, as for the shape. That is, we decompose

$$dw^2 + w^2 d\Omega_3^2 = d\rho^2 + dL^2 + \rho^2 d\Omega_3^2 + L^2 d\psi^2, \quad (3.58)$$

where $w^2 = \rho^2 + L^2$, and the embedding function is given by $L = L(\rho)$ and $\psi = \text{const.}$

Some calculations of the Lagrangian lead to the following expression for the action:

$$S = \text{const.} \int d^4x \rho^3 \sqrt{\xi (1 + (L')^2)} \quad (3.59)$$

where

$$\begin{aligned} \xi \equiv & \left(\frac{w^4 + r_H^4/4}{w^4} \right)^4 h(w) \\ & - \left(\frac{2\pi\alpha' R^2 (w^4 + r_H^4/4)}{w^6} \right)^2 (\mathbf{E}^2 - h(w) \mathbf{B}^2) - \left(\frac{2\pi\alpha' R^2}{w^2} \right)^4 (\mathbf{E} \cdot \mathbf{B})^2. \end{aligned} \quad (3.60)$$

By increasing E_i in this expression, there exists a critical electric field at which $\xi = 0$. In other words, for fixed B_i , E_i , and the background (temperature T), there exists L at which $\xi = 0$ is realized. Let us denote that value of L as L_0 . Then, expanding the scalar function $L(\rho)$ around $L = L_0$ as

$$L(\rho) = L_0 + f(\rho), \quad (3.61)$$

we can have a leading-order action

$$S = \text{const.} \int d^4x \rho^3 \sqrt{f (1 + (f')^2)}, \quad (3.62)$$

which is exactly of the form (3.34) with $p = 5$. Therefore, we again obtain a conic D-brane whose tip is at $L = L_0$:

$$L(\rho) = L_0 + \frac{1}{\sqrt{6}}\rho + \text{higher}. \quad (3.63)$$

3.3.2. Force balance of the cone

Let us study the force balance. The induced metric on the D7-brane becomes

$$h_{ab} dy^a dy^b = \frac{1}{R^2} \frac{w^4 + r_H^4/4}{w^2} \left[-h(w) dt^2 + dx_i^2 \right] + \frac{R^2}{w^2} \left[(1 + L'^2) d\rho^2 + \rho^2 d\Omega_3^2 \right]. \quad (3.64)$$

The unit vector and 1-form normal to the brane are given by

$$n^\mu \partial_\mu = \frac{w}{R} \frac{1}{\sqrt{1+L'^2}} [L' \partial_\rho - \partial_L], \quad n_\mu dx^\mu = \frac{R}{w} \frac{1}{\sqrt{1+L'^2}} [L' d\rho - dL], \quad (3.65)$$

and the unit vector and 1-form tangent to the brane are

$$r^\mu \partial_\mu = \frac{w}{R} \frac{1}{\sqrt{1+L'^2}} [\partial_\rho + L' \partial_L], \quad r_\mu dx^\mu = \frac{R}{w} \frac{1}{\sqrt{1+L'^2}} [d\rho + L' dL]. \quad (3.66)$$

The extrinsic curvature for the direction with the normal vector n^μ is

$$\begin{aligned} K^0_0 &= \frac{1}{Rw} \left(\sqrt{h} + \frac{w}{2h} \frac{dh}{dw} \right) \frac{\rho L' - L}{\sqrt{1+L'^2}}, \\ K^i_j &= \frac{\sqrt{h}(w)}{Rw} \frac{\rho L' - L}{\sqrt{1+L'^2}} \delta^i_j, \\ K^m_n &= \frac{1}{Rw} \frac{\rho + LL'}{\sqrt{1+L'^2}} \frac{L}{\rho} \delta^m_n, \end{aligned} \quad (3.67)$$

and

$$K_{\mu\nu} r^\mu r^\nu = \frac{w^2}{R} \left(\frac{L'}{w\sqrt{1+L'^2}} \right)' + \frac{L\sqrt{1+L'^2}}{Rw}. \quad (3.68)$$

The non-vanishing components of the stress–energy tensor of the D7-brane are

$$\begin{aligned} T^0_0 &= -\frac{1}{\sigma} \left(1 + \frac{\mathbf{B}^2}{g^2} \right), \quad T^0_i = \frac{1}{\sigma} \frac{1}{g^2 h} \epsilon_{ijk} E^j B^k, \quad T^i_0 = -\frac{1}{\sigma} \frac{1}{g^2} \epsilon^{ijk} E_j B_k, \\ T^i_j &= -\frac{1}{\sigma} \left[\left(1 - \frac{\mathbf{E}^2}{g^2 h} \right) \delta^i_j + \frac{1}{g^2} (h^{-1} E^i E_j + B^i B_j) \right], \\ T^m_n &= -\sigma \delta^m_n, \quad T_{\mu\nu} r^\mu r^\nu = -\sigma, \end{aligned} \quad (3.69)$$

where

$$\sigma^2 \equiv 1 - \frac{1}{g^2} (h^{-1} \mathbf{E}^2 - \mathbf{B}^2) - \frac{1}{g^4 h} (\mathbf{E} \cdot \mathbf{B})^2, \quad (3.70)$$

and

$$g(w) \equiv \frac{1}{2\pi\alpha' R^2} \frac{w^4 + r_H^4/4}{w^2}. \quad (3.71)$$

They imply that, if the electric field becomes sufficiently large to be $\sigma = 0$, some components of the stress–energy tensor, meaning the isotropic tension, will vanish and others will diverge.

From Eq. (2.12), the equation of motion for the brane can be written as

$$T^0_0 K^0_0 + T^i_j K^j_i + T^m_n K^n_m + T^{\mu\nu} K_{\alpha\beta} r_\mu r_\nu r^\alpha r^\beta = 0. \quad (3.72)$$

Now, we suppose that the brane shape will become conical at the critical point, namely, $\sigma = 0$ at $\rho = 0$. Since $T_{\mu\nu} r^\mu r^\nu$ vanishes at $\sigma = 0$, the equation of motion reduces to

$$T^0_0 K^0_0 + T^i_j K^j_i \simeq -T^m_n K^n_m, \quad (3.73)$$

which means force balance at the tip of the cone. It turns out that the stress–energy tensor on the left-hand side will diverge at $\sigma = 0$ while the extrinsic curvature on the right-hand side will diverge because of $\rho = 0$. In contrast to the previous two examples, there is now no explicit external force. However, gravitational force due to the bulk AdS space is acting on the brane and balances with

the tension coupling the extrinsic curvature of the cone. By using $\sigma(\rho=0)=0$, Eq. (3.73) can be explicitly rewritten as

$$-\frac{1}{R} \frac{d\sigma}{dw} \frac{\rho L' - L}{\sqrt{1+L'^2}} \simeq 3\sigma \frac{1}{Rw} \frac{\rho + LL'}{\sqrt{1+L'^2}} \frac{L}{\rho}. \quad (3.74)$$

As a result, we have

$$L'|_{\rho=0} = \frac{1}{\sqrt{6}}, \quad (3.75)$$

where we have used $\sigma/\rho|_{\rho=0} = 2w'd\sigma/dw|_{\rho=0}$ and $w|_{\rho=0} = L_0$. This is equivalent to the conic solution (3.63).

3.4. Universal cone angle

The Taylor cones have a universal cone angle. We have seen that the mechanism of the formation of conic D-branes is quite similar to that of Taylor cones; thus, we expect that there may exist a universal cone angle for the D-brane cones.

In fact, we find that the half-cone angle is universally determined as

$$\theta_{\text{cone}} = \arctan \sqrt{2(d_{\text{cone}} - 1)}, \quad (3.76)$$

where d_{cone} is the dimension of the cone (in other words, the cone is $\mathbf{R}_+ \times S^{d_{\text{cone}}-1}$). To show this, we first look at the example in the NSNS background in Sect. 3.2. There the Dp -brane has the cone in the worldvolume directions (x^2, x^3, \dots, x^p) , so the cone is $(p-1)$ -dimensional: $d_{\text{cone}} = p-1$. From the cone solution (3.38), the half-cone angle defined as

$$\tan \theta_{\text{cone}} = \left(\frac{d\phi}{d\rho} \right)^{-1} \Big|_{\rho=0} \quad (3.77)$$

is given by (3.76).

This angle (3.76) of Sect. 3.2 should be quite universal, since the linear part of the solution (3.38) does not depend on the value c of the background NSNS flux. Let us check the universality of the cone angle below.

For the D-brane cone in the RR background in Sect. 3.1, the dimension of the cone is obviously $d_{\text{cone}} = 2$, so the formula (3.76) suggests $\theta_{\text{cone}} = \arctan \sqrt{2}$. This coincides with the solution (3.15) in the RR background. Again, the angle does not depend on the strength of the RR flux, so the angle is universal.

Furthermore, as for the D7-brane cone in AdS in Sect. 3.3, the cone angle is found to be

$$\theta_{\text{cone}} = \arctan \sqrt{6}. \quad (3.78)$$

The probe D7-brane has 4 spatial dimensions for its worldvolume in the extra dimensions, since the worldvolume along (x^0, \dots, x^3) is assumed to be flat as a quark flavor D-brane. So the current situation corresponds to the case of $d_{\text{cone}} = 4$. Therefore this (3.78) coincides again with the universal cone angle formula (3.76). Again, the cone angle of the D7-brane in AdS is independent of all the background values: the black hole temperature T , the magnetic field B_i , the critical electric field E_i , and the position of the cone tip L_0 .

Note that, in the last case, the angle, of course, should be given by the inner product associated with the bulk metric between tangent vectors on the brane worldvolume,⁸ so that the value does not depend on the choice of coordinates. Now, the metric (3.53) that we use is conformal to Euclidean space in a Cartesian coordinate system for the extra dimensions, as seen from the factor $(dw^2 + w^2 d\Omega_5^2)$. The expression of the angle is the same as that in flat space.

From all the examples that we have studied above, the half-cone angle (3.76) is independent of the background parameters. In comparison with the general formula (2.28) discussed in Sect. 2.2, a factor of $(d_{\text{cone}} - 1)$ obviously comes from the dimension of the spherical part of the cone, and the other factor of two is related to the power of the stress distribution on the cone, i.e., the tension on the cone behaves as r^α with $\alpha = 1/2$ at a distance r from the apex. We conjecture that the D-brane cone angle (3.76) is independent of anything related to the background fields. This universality is reminiscent of the Taylor cones.

4. Universality of spectra in AdS/CFT

4.1. Observables in boundary theory

In Sect. 3.3, we have seen that the cone angle is universal for the probe D7-brane in the $\text{AdS}_5\text{--Schwarzschild} \times S^5$ spacetime: The cone angle does not depend on the parameters, Hawking temperature T , electric field E , and magnetic field B , once we impose the critical condition on the tip of the brane. The D3/D7 system is dual to $\mathcal{N} = 2$ supersymmetric QCD. Here, we will investigate how we can observe the universality of the cone angle in view of dual gauge theory. A static D7-brane solution in the bulk spacetime is written as $L = L(\rho)$. Near the AdS boundary $\rho = \infty$, the solution is expanded as

$$L(\rho) = L_\infty + \frac{c}{\rho^2} + \cdots \quad (4.1)$$

The constants L_∞ and c correspond to quark mass m_q and quark condensate $\langle \mathcal{O} \rangle$ as

$$m_q = \left(\frac{\lambda}{2\pi^2} \right)^{1/2} \frac{L_\infty}{R^2}, \quad (4.2)$$

$$\langle \mathcal{O} \rangle = -\frac{N_c \sqrt{\lambda}}{2^{3/2} \pi^3} \frac{c}{R^6}. \quad (4.3)$$

The other observable in the boundary theory is the energy density. The quark condensate and energy density are summations of contributions from all meson excitations. To obtain expressions for each meson excitation, we will study the linear perturbation theory in the following subsections. We will find that the universality of the cone angle is observed as the universality of the spectra of quark condensate and energy density.

We are also motivated by the “turbulent meson condensation” proposed in Refs. [7,8]. We have numerically studied the time evolution of energy spectra in dynamical and quasi-static processes in the D3/D7 system and found that, as far as we investigated, the energy spectrum always obeys the power law, $\varepsilon_n \sim n^{-5}$, at the phase boundary of the black hole and Minkowski embeddings. So, we conjectured that the exponent -5 in the energy spectrum is universal for phase transitions in $\mathcal{N} = 2$

⁸ The angle between vectors u and v on the metric g is defined by $\theta(u, v) = \arccos \frac{g(u, v)}{\sqrt{g(u, u)}\sqrt{g(v, v)}}$ where $g(u, v) \equiv u^\mu v^\nu g_{\mu\nu}$. The angle and the length are geometrically independent quantities, because, e.g., under a conformal transformation $g \rightarrow \Omega^2 g$ the angle is invariant but the length is not.

SQCD (super QCD). We will give an analytic derivation of the exponent for a quasi-static process in this section.

4.2. Linear perturbation theory

For the zero temperature $r_H = 0$, the background metric (3.53) reduces to the $\text{AdS}_5 \times S^5$ spacetime:

$$ds^2 = \frac{w^2}{R^2} (-dt^2 + dx_i^2) + \frac{R^2}{w^2} (d\rho^2 + \rho^2 d\Omega_3^2 + dL^2 + L^2 d\psi^2), \quad (4.4)$$

where $w^2 = \rho^2 + L^2$. We impose the spherical symmetry of S^3 and translation invariance ∂_{x_i} on the D7-brane. Then, the dynamics of the D7-brane in this spacetime is described by a single function $L = L(t, \rho)$. The static solution is trivially given by a constant: $L(t, \rho) = L_\infty$. We consider linear perturbation of the static solution. Defining the perturbation variable $\delta L = L(t, \rho) - L_\infty$, we obtain the second-order action for δL as

$$S = \pi^2 \mathcal{T}_7 R^4 V_3 \int dt d\rho \frac{\rho^3}{(\rho^2 + L_\infty^2)^2} \left[\delta \dot{L}^2 - \frac{(\rho^2 + L_\infty^2)^2}{R^4} \delta L'^2 \right], \quad (4.5)$$

where $\dot{} = \partial_t$, $' = \partial_\rho$, $V_3 = \int dx_1 dx_2 dx_3$, and $\mathcal{T}_7 = (2\pi)^{-7} \alpha'^{-4} g_s$. The equation of motion for δL is

$$\left(\partial_t^2 + \mathcal{H} \right) \delta L = 0, \quad \mathcal{H} = -\frac{(\rho^2 + L_\infty^2)^2}{R^4 \rho^3} \partial_\rho \rho^3 \partial_\rho. \quad (4.6)$$

The operator \mathcal{H} is Hermitian under an inner product:

$$(\alpha, \beta) = \int_0^\infty d\rho \frac{\rho^3}{(\rho^2 + L_\infty^2)^2} \alpha \beta. \quad (4.7)$$

We define the norm using the inner product as $\|\alpha\|^2 = (\alpha, \alpha)$. Eigenvalues ω_n^2 and eigenfunctions e_n of \mathcal{H} are given by

$$e_n = \mathcal{N}_n \frac{L_\infty^2}{\rho^2 + L_\infty^2} F\left(n+3, -n, 2; \frac{L_\infty^2}{\rho^2 + L_\infty^2}\right), \quad \omega_n^2 = 4(n+1)(n+2) \frac{L_\infty^2}{R^4}. \quad (4.8)$$

where $n = 0, 1, 2, \dots$ and $\mathcal{N}_n = \sqrt{2(2n+3)(n+1)(n+2)}$. The eigenfunctions are normalized as $(e_n, e_m) = \delta_{nm}$. We expand the perturbation variable as

$$\delta L = \sum_{n=0}^{\infty} c_n(t) e_n(\rho). \quad (4.9)$$

The asymptotic form of the eigenfunction is $e_n \simeq \mathcal{N}_n L_\infty^2 / \rho^2$ ($\rho \rightarrow \infty$). Thus, from Eq. (4.3), the quark condensate for the fluctuating D7-brane is written as

$$\langle \mathcal{O} \rangle = \sum_{n=0}^{\infty} \langle \mathcal{O}_n \rangle, \quad \langle \mathcal{O}_n \rangle = -\frac{\mathcal{N}_n N_c m_q^3}{\lambda} \frac{c_n(t)}{L_\infty}, \quad (4.10)$$

where $\langle \mathcal{O}_n \rangle$ can be regarded as the quark condensate contributed from the n th excited mesons. We eliminated the AdS curvature scale R using Eq. (4.2). So, the quark mass m_q appears in the expression

of $\langle \mathcal{O}_n \rangle$. From the second-order action (4.5), we obtain the energy density as

$$\varepsilon = \pi^2 T_7 R^4 \int d\rho \frac{\rho^3}{(\rho^2 + L_\infty^2)^2} \left[\delta \dot{L}^2 + \frac{(\rho^2 + L_\infty^2)^2}{R^4} \delta L'^2 \right]. \quad (4.11)$$

Substituting Eq. (4.9) into the above expression, we have

$$\varepsilon = \sum_{n=0}^{\infty} \varepsilon_n, \quad \varepsilon_n = \frac{N_c m_q^2}{8\pi^2 L_\infty^2} (\dot{c}_n^2 + \omega_n^2 c_n^2). \quad (4.12)$$

We substituted $\alpha'^2 = R^4 / (2\lambda)$ and eliminated R using Eq. (4.2) again. ε_n can be regarded as the energy contribution from the n th excited meson.

4.3. Spectra in nonlinear theory

We extend the definitions of spectra $\langle \mathcal{O}_n \rangle$ and ε_n to nonlinear theory. Denoting a static nonlinear D7-brane solution in the $\text{AdS}_5\text{--Schwarzschild} \times S^5$ spacetime as $L = L(\rho)$, we define the spectra of the quark condensate and energy density for the nonlinear solution as

$$\langle \mathcal{O}_n \rangle = -\frac{\mathcal{N}_n N_c m_q^3}{\lambda L_\infty} (L - L_\infty, e_n), \quad \varepsilon_n = \frac{N_c m_q^2 \omega_n^2}{8\pi^2 L_\infty^2} (L - L_\infty, e_n)^2, \quad (4.13)$$

where we have omitted the time derivative of the mode coefficient \dot{c}_n in Eq. (4.11) since we consider only static solutions in nonlinear theory. We will see that the universality of the cone angle can be seen as the universality of spectra in the large- n limit.

For the purpose of studying the large- n behavior of the spectra, it is convenient to show the following lemma:

Lemma 1. *Let $f(\rho)$ be a smooth function in $\rho \in (0, \infty)$ whose norm is finite, i.e., $\|f\|^2 < \infty$.⁹ Then,*

$$(f, e_n) \rightarrow 0, \quad (n \rightarrow \infty). \quad (4.14)$$

Proof From the positivity of the norm, we obtain

$$\begin{aligned} 0 &\leq \left\| f - \sum_{n=0}^N (f, e_n) e_n \right\|^2 \\ &= \|f\|^2 - 2 \sum_n (f, e_n)^2 + \sum_{n,m} (f, e_n) (f, e_m) (e_n, e_m) = \|f\|^2 - \sum_n (f, e_n)^2. \end{aligned} \quad (4.15)$$

At the last equality, we used the orthonormal relation $(e_n, e_m) = \delta_{nm}$. Hence, we have

$$\sum_{n=0}^N (f, e_n)^2 \leq \|f\|^2. \quad (4.16)$$

This inequality is known as Bessel's inequality. Now, the right-hand side of the inequality is finite and does not depend on N . Therefore, (f, e_n) must decrease to zero for $n \rightarrow \infty$. ■

In the Fourier analysis, this is well known as the Riemann–Lebesgue lemma.

⁹ The condition for the finiteness of the norm can be explicitly written as $f = o(1/\rho^2)$ ($\rho \rightarrow 0$) and $f = o(1)$ ($\rho \rightarrow \infty$), where o is Landau's little- o notation.

4.4. Spectra of non-conical solutions

Firstly, we investigate the large- n behavior of the spectra for non-conical solutions. Solving the equation of motion derived from the DBI action (3.59) near the axis $\rho = 0$, the non-conical solution is expanded as

$$L(\rho) = a_0 + a_2 \rho^2 + a_4 \rho^4 + \cdots, \quad (\rho \sim 0). \quad (4.17)$$

On the other hand, at infinity, the solution behaves as

$$L(\rho) - L_\infty = \frac{b_2}{\rho^2} + \frac{b_4}{\rho^4} + \frac{b_6}{\rho^6} + \cdots, \quad (\rho \sim \infty). \quad (4.18)$$

Operating \mathcal{H} to the nonlinear solution m times, we have

$$\mathcal{H}^m L = \mathcal{O}(1), \quad (\rho \sim 0), \quad \mathcal{H}^m L = \mathcal{O}(1/\rho^2), \quad (\rho \sim \infty). \quad (4.19)$$

So, $\|\mathcal{H}^m L\|^2$ is finite for any m . Therefore, from Eq. (4.14), we obtain

$$(\mathcal{H}^m L, e_n) = (L - L_\infty, \mathcal{H}^m e_n) = \omega_n^{2m} (L - L_\infty, e_n) \rightarrow 0, \quad (n \rightarrow \infty). \quad (4.20)$$

At the first equality, we used the hermiticity of \mathcal{H} . This implies that $(L - L_\infty, e_n)$ must fall off faster than $1/\omega_n^{2m} \sim n^{-2m}$ as a function of n . Since m is any integer, $(L - L_\infty, e_n)$ must fall off faster than any power-law functions of n . Therefore, the spectra of the quark condensate and energy density also fall off faster than any power. Actually, by the numerical calculation, it is suggested that the spectra fall off exponentially for non-conical solutions [7,8].

4.5. Spectra of conical solutions

Now, we consider the mode decomposition of a conical solution $L = L_c(\rho)$. Near the axis $\rho = 0$, the critical solution is expanded as

$$L_c(\rho) = a_0 + \cot \theta_{\text{cone}} \rho + a_2 \rho^2 + a_3 \rho^3 + \cdots, \quad (\rho \sim 0), \quad (4.21)$$

where θ_{cone} represents the cone angle. The asymptotic form at infinity is the same as that for the non-conical solution (4.18). We define \tilde{L}_c , which satisfies the Neumann condition at $\rho = 0$, as

$$L_c(\rho) = \tilde{L}_c(\rho) + L_\infty S(\rho) \cot \theta_{\text{cone}}. \quad (4.22)$$

where we define the function S as

$$S(\rho) \equiv \frac{L_\infty^2 \rho}{(\rho^2 + L_\infty^2)^{3/2}}, \quad (4.23)$$

We chose this function so that we can carry out its mode decomposition analytically as¹⁰

$$\begin{aligned} (S, e_n) &= \frac{6(-1)^{n+1} \sqrt{2(2n+3)(n+1)(n+2)}}{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)} \\ &\simeq \frac{3}{8} (-1)^{n+1} n^{-7/2}, \quad (n \rightarrow \infty). \end{aligned} \quad (4.24)$$

Next we consider the mode decomposition of $\tilde{L}_c(\rho)$. The asymptotic form of \tilde{L}_c becomes

$$\tilde{L}_c(\rho) = a_0 + a_2 \rho^2 + \left(a_3 + \frac{3}{2L^2} \cot \theta_{\text{cone}} \right) \rho^3 + \cdots, \quad (\rho \sim 0). \quad (4.25)$$

¹⁰ We expressed the eigenfunction e_n by the series of $L_\infty^2/(\rho^2 + L_\infty^2)$ and integrated termwise.

There is no linear term in ρ since it is subtracted by $L_\infty S(\rho) \cot \theta_{\text{cone}}$ in Eq. (4.22). Operating \mathcal{H} twice to $\tilde{L}_c(\rho)$, we obtain

$$\mathcal{H}^2 \tilde{L}_c = \mathcal{O}(1/\rho), \quad (\rho \sim 0), \quad \mathcal{H}^2 \tilde{L}_c = \mathcal{O}(1/\rho^2), \quad (\rho \sim \infty). \quad (4.26)$$

Although $\mathcal{H}^2 \tilde{L}_c$ diverges at $\rho = 0$, its norm is still finite. (Because of the measure in the inner product (4.7), the integrand is regular at $\rho = 0$.) From Eq. (4.14), we obtain

$$(\mathcal{H}^2 \tilde{L}_c, e_n) = \omega_n^4 (\tilde{L}_c - L_\infty, e_n) \rightarrow 0, \quad (n \rightarrow \infty). \quad (4.27)$$

So, $(\tilde{L}_c - L_\infty, e_n)$ falls off faster than $1/n^4$ and is a subleading term in the limit of $n \rightarrow \infty$. Thus, we have

$$(L_c - L_\infty, e_n) \simeq \frac{3}{8} (-1)^{n+1} n^{-7/2} L_\infty \cot \theta_{\text{cone}}, \quad (n \rightarrow \infty). \quad (4.28)$$

Therefore, from Eq. (4.13), the spectra of the quark condensate and energy density become

$$\langle \mathcal{O}_n \rangle \simeq \frac{3N_c m_q^3 \cot \theta_{\text{cone}}}{4\lambda} \frac{(-1)^n}{n^2}, \quad \varepsilon_n \simeq \frac{9N_c m_q^4 \cot^2 \theta_{\text{cone}}}{64\lambda} \frac{1}{n^5}, \quad (n \rightarrow \infty). \quad (4.29)$$

There are some remarks to make on the large- n limit of the spectra. The spectrum of the quark condensate is positive/negative when n is an even/odd number. This is one of the simplest predictions from the AdS/CFT calculation of the spectrum. The power of the spectra does not depend on the cone angle θ_{cone} . So, the universality of the cone angle has nothing to do with the universality of the power in the spectra: If the D7-brane has a cone, we always have $\langle \mathcal{O}_n \rangle \propto n^{-2}$ and $\varepsilon_n \propto n^{-5}$ for $n \rightarrow \infty$. The universality of the cone angle, however, appears as the universality of the coefficient of the power-law behavior in the spectra. This could be one of the observable effects in the dual gauge theory. It would be nice if these predictions from AdS/CFT could be confirmed by lattice QCD.

5. Conclusion and discussion

In this paper, we have found conic D-brane solutions under external uniform RR or NSNS field strengths. The apex angle is found to be universal. The angle formula (1.1) depends only on the dimensions of the cone.

As we have emphasized, the universal angle of the D-brane cone is similar to the one for Taylor cones in fluid mechanics. The D-brane mechanics is governed by the DBI action, which normally exhibits a form such as (3.13) when the tension goes to zero at a certain point on the worldvolume. This special form (3.13) is important for having the conic shape and the universal angle.

When D-branes touch an event horizon in the target space, a cone forms, which is nothing but the critical embedding at thermal phase transition in AdS/CFT. However, the situation is different in our cases. For example, the special form (3.13) consists of an overall factor \sqrt{f} related to the tension and the Nambu–Goto part $\sqrt{1 + (f')^2}$ simply coming from the volume element of the brane. For a thermal phase transition [4,5], the conic shape appears due to the nontrivial background spacetime and the deformation of the Nambu–Goto part other than the tension. For our cases, the background geometry does not help, while the coupling to the flux generates the vanishing factor \sqrt{f} , which results in cancellation of the tension. In fact, in general, for DBI actions, the whole Lagrangian is written with a square root, so when the factor goes to zero, it naturally vanishes as \sqrt{f} for $f \rightarrow 0$. This dependence determines the power of the distribution of the tension as $\alpha = 1/2$ in (2.9) and thus leads to the factor 2 in the apex angle formula (1.1). This plausible argument supports the genericity of the angle formula (1.1) for any D-brane cones caused by external uniform forces.

The vanishing stress of the conic D-brane at its apex is similar to D-brane supertubes [14] and tachyon condensation [15]. In both cases, peculiar dispersions for propagation modes have been reported [16,17]. It would be interesting to study the modes around the apex of our conic D-branes. One possible obstacle would be higher-derivative corrections. In our examples, since the apex angle is universal, it cannot be gradually changed. So the effects of the higher-derivative terms can be non-infinitesimal.

In a sense, our example of the D2–D0 cone in the RR 2-form flux contrasts a renowned Myers effect [18] for D2–D0 in an RR 4-form flux, the dielectric branes. The Myers effect is a dielectric polarization of a D2-brane that is caused by the 4-form flux pulling the D2-brane surface. In our case, the 2-form flux pulls the bound D0-brane on the surface of the D2-brane, so the mechanism of the forces is different, and, resultantly, the shape of the D2-brane is different: a spherical or a conic shape. It is obvious that these two effects can be combined once we allow both the 2-form and the 4-form. More complicated brane shapes can emerge, which may be important for various applications of D-brane physics.

Finally, we would like to make a comment on spiky singularities in membrane quantization [1]. We are not sure if the conic D-brane configurations may help to resolve the issue or not. However, one important observation is that, since the apex angle is unique, the conic brane configuration can exist even if we gradually turn off the background flux. This fact may signal a possible classification of classical D-brane configurations. We hope that this direction of research may help with the quantization problem.

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Appendix A. Stress–energy tensor of D-branes

We consider a D p -brane in d -dimensional spacetime. Let $\{x^\mu\}$ and $\{y^a\}$ be coordinates on the bulk spacetime and the brane worldvolume, respectively. The brane is characterized by embedding functions $x^\mu = X^\mu(y)$. The DBI action for the D p -brane is given by

$$\begin{aligned} S_{Dp} &= - \int d^{p+1}y \sqrt{-\det(h_{ab} + F_{ab})} \\ &= - \int d^{p+1}y \sqrt{-\det h_{ab}} \sigma, \end{aligned} \quad (\text{A1})$$

where

$$\sigma^2 \equiv \sum_{k=0}^{\lfloor (p+1)/2 \rfloor} F_{a_1}{}^{b_1} \cdots F_{a_{2k}}{}^{b_{2k}} \delta_{[b_1}{}^{a_1} \cdots \delta_{b_{2k}}]{}^{a_{2k}}. \quad (\text{A2})$$

Note that h_{ab} denotes the induced metric, which is used for raising and lowering Latin indices. If the rank of F_{ab} is less than or equal to three, we have explicitly $\sigma^2 = 1 + \frac{1}{2} F_{ab} F^{ab}$. If the rank of F_{ab} is less than or equal to five, we have $\sigma^2 = 1 + \frac{1}{2} F_{ab} F^{ab} - \frac{1}{4} F_{a_1}{}^{a_2} F_{a_2}{}^{a_3} F_{a_3}{}^{a_4} F_{a_4}{}^{a_1} + \frac{1}{8} (F_{ab} F^{ab})^2$.

The stress–energy tensor of the Dp -brane in the bulk is given by

$$\begin{aligned}\hat{T}^{\mu\nu} &= \frac{2}{\sqrt{-g}} \frac{\delta S_{Dp}}{\delta g_{\mu\nu}} \\ &= -\frac{1}{\sqrt{-g}} \int d^{p+1}y \sqrt{-h} \sigma \left(\gamma^{-1} \right)^{ab} \partial_a X^\mu \partial_b X^\nu \delta^{(d)}(x - X(y)),\end{aligned}\quad (\text{A3})$$

where $(\gamma^{-1})^{ab}$ denotes the inverse of the effective metric defined by $\gamma_{ab} \equiv h_{ab} + F_{ac}F_{bd}h^{cd}$. Here, we introduce $(d - p - 1)$ functions $z^i(x)$ in the bulk such that $z^i(X(y)) = z_0^i$ are satisfied, where arbitrary constants z_0^i can be zero without loss of generality. We have a diffeomorphism $x^\mu = \varphi^\mu(y^a, z^i)$ such that $X^\mu(y^a) = \varphi^\mu(y^a, z^i = 0)$. This means that the neighborhood of the brane is locally spanned by new bulk coordinates $\{y^a, z^i\}$ and the submanifold of the brane is characterized by $z^i = 0$. The bulk metric can be rewritten as

$$g_{\mu\nu} dx^\mu dx^\nu = h_{ab} dy^a dy^b + N_{ij} dz^i dz^j. \quad (\text{A4})$$

As a result, we have

$$\begin{aligned}\hat{T}^{\mu\nu} &= -\frac{1}{\sqrt{-g}(x)} \int d^{p+1}y \sqrt{-h} \sigma \left(\gamma^{-1} \right)^{ab} \partial_a X^\mu \partial_b X^\nu \delta^{(d)}(x - X(y)) \\ &= -\frac{1}{\sqrt{N}(y', z')} \int d^{p+1}y \sigma \left(\gamma^{-1} \right)^{ab} \partial_a X^\mu \partial_b X^\nu \delta^{(p+1)}(y' - y) \delta^{(d-p-1)}(z') \\ &= -\frac{1}{\sqrt{N}(y', z')} \sigma \left(\gamma^{-1} \right)^{ab} \partial_a X^\mu \partial_b X^\nu \delta^{(d-p-1)}(z'),\end{aligned}\quad (\text{A5})$$

where we have used $\delta^{(d)}(x)/\sqrt{-g} = \delta^{(p+1)}(y) \delta^{(d-p-1)}(z)/\sqrt{-hN}$ and $N \equiv \det N_{ij}$.

Now, the stress–energy tensor localized on the brane is obtained by integrating $\hat{T}^{\mu\nu}$ along the directions perpendicular to the brane as

$$\begin{aligned}T^{ab} &= \int d^{d-p-1}z \sqrt{N} \hat{T}^{\mu\nu} h^a_\mu h^b_\nu \\ &= -\sigma \left(\gamma^{-1} \right)^{ab}.\end{aligned}\quad (\text{A6})$$

It is worth noting that this is equivalent to the stress–energy tensor derived from variation of the induced metric on the worldvolume as

$$\begin{aligned}T^{ab} &= \frac{2}{\sqrt{-h}} \frac{\delta S_{Dp}}{\delta h_{ab}} \\ &= -2 \frac{\delta \sigma}{\delta h_{ab}} - \sigma h^{ab} = -\sigma \left(\gamma^{-1} \right)^{ab}.\end{aligned}\quad (\text{A7})$$

This expression makes it clear that, since σ is not constant but depends on the induced metric for DBI branes, the energy density is not equal to the tension (i.e., negative pressure) unlike in the case of Nambu–Goto branes.

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